Probability in High Dimensions

Solutions to Exercises, Week 5

August 14, 2020

2.4 Variance identities and exponential ergodicity

Problem 2.11 (Covariance identities)**.** *Let P_t be a reversible ergodic Markov semigroup with stationary measure μ.*

a. Prove the following identity:

$$\mathrm{Cov}_\mu(f,g) = 2\int_0^\infty \mathcal{E}(P_tf, P_tg)dt.$$

b. Prove the following identity:

$$\mathrm{Cov}_\mu(f,g) = \int_0^\infty \mathcal{E}(f, P_tg)dt.$$

c. Let $X \sim N(0,\Sigma)$ be a random vector with each of the entries having positive correlation, i.e. $\Sigma_{ij} \geq 0$ for all i,j. Prove that this implies the following much strong positive association *property: for every pair of functions f,g that are coordinatewise increasing, we have* $\mathrm{Cov}(f(X),g(X)) \geq 0$.

Solution. a) Since $\mathcal{L}$ is self-adjoint

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Cov}\left(P_tf,P_tg\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\langle P_tf, P_tg\rangle \\ &= \langle \mathcal{L}P_tf, P_tg\rangle + \langle P_tf, \mathcal{L}P_tg\rangle \\ &= 2\langle P_tf, \mathcal{L}P_tg\rangle = -2\mathcal{E}(P_tf,P_tg).\end{aligned}$$

Now, since P_t is ergodic,

$$\begin{aligned}\mathrm{Cov}(f,g) &= \mathrm{Cov}(P_0f,P_0g) - \lim_{t\to\infty}\mathrm{Cov}(P_tf,P_tg) \\ &= -\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Cov}\left(P_tf,P_tg\right)\mathrm{d}t \\ &= 2\int_0^\infty \mathcal{E}(P_tf,P_tg)\mathrm{d}t\end{aligned}$$

b) We have

$$\begin{aligned}-\mathcal{E}(P_tf,P_tg) &= \langle P_tf, \mathcal{L}P_tg\rangle \\ &= \langle f, P_t\mathcal{L}P_tg\rangle \\ &= \langle f, \mathcal{L}P_{2t}g\rangle = -\mathcal{E}(f,P_{2t}g)\end{aligned}$$

since P_t is self-adjoint and $\mathcal{L}$ and P_t commute. A change of variable yields

$$2\int_0^\infty \mathcal{E}(P_tf,P_tg)\mathrm{d}t = \int_0^\infty \mathcal{E}(f,P_tg)\mathrm{d}t.$$

c) Let $X \sim \mathcal{N}(0, \Sigma)$ be a centred Gaussian vector in $\mathbb{R}^n$ with covariance matrix Σ. Assume that $\Sigma_{i,j} > 0$ for all $i.j$. Then for any two coordinate-wise increasing functions f, g we have that

$$\begin{aligned}\mathrm{Cov}_\mu(f,g) &= \mathbb{E}[f(X)g(X)] - \mathbb{E}[f(X)]\mathbb{E}[g(X)] \\ &= \mathbb{E}[f(\Sigma^{1/2}Y)g(\Sigma^{1/2}Y)] - \mathbb{E}[f(\Sigma^{1/2}Y)]\mathbb{E}[g(\Sigma^{1/2}Y)] \\ &= \mathrm{Cov}_\nu[\tilde{f}, \tilde{g}],\end{aligned}$$

where $\nu = \mathcal{N}(0, I)$, $\tilde{f}(\cdot) = f(\Sigma^{1/2}\cdot)$ and $\tilde{g}(\cdot) = g(\Sigma^{1/2}\cdot)$. Then

$$\begin{aligned}&\mathrm{Cov}_\nu[\tilde{f}, \tilde{g}] \\ &= \int_0^\infty \mathrm{d}t \mathcal{E}(P_t\tilde{f}, P_t\tilde{g}) \\ &= \int_0^\infty \mathrm{d}t \mathbb{E}_\nu \left[\nabla P_t\tilde{f}(\xi) \cdot \nabla P_t\tilde{g}(\xi)\right] \\ &= \int_0^\infty \mathrm{d}t \int \nu(\mathrm{d}x) \nabla \int \nu(\mathrm{d}\psi) \tilde{f}\left(\mathrm{e}^{-t}x + \sqrt{1-\mathrm{e}^{-2t}}\psi\right) \\ &\quad \cdot \nabla \int \nu(\mathrm{d}\xi) \tilde{f}\left(\mathrm{e}^{-t}x + \sqrt{1-\mathrm{e}^{-2t}}\psi\right) \tilde{g}\left(\mathrm{e}^{-t}x + \sqrt{1-\mathrm{e}^{-2t}}\xi\right) \\ &= \int_0^\infty \mathrm{d}t \int \nu(\mathrm{d}x) \int \nu(\mathrm{d}\xi) \nabla\tilde{f}\left(\mathrm{e}^{-t}x + \sqrt{1-\mathrm{e}^{-2t}}\psi\right) \cdot \nabla\tilde{g}\left(\mathrm{e}^{-t}x + \sqrt{1-\mathrm{e}^{-2t}}\xi\right) \\ &= \int_0^\infty \mathrm{d}t \int \nu(\mathrm{d}x) \int \nu(\mathrm{d}\xi) \int \nu(\mathrm{d}\psi) \nabla f\left(\mathrm{e}^{-t}\Sigma^{1/2}x + \sqrt{1-\mathrm{e}^{-2t}}\Sigma^{1/2}\psi\right) \\ &\quad \cdot \nabla g\left(\mathrm{e}^{-t}\Sigma^{1/2}x + \sqrt{1-\mathrm{e}^{-2t}}\Sigma^{1/2}\xi\right).\end{aligned}$$

Then notice that

$$\frac{\partial}{\partial x_i} f\left(\mathrm{e}^{-t}\Sigma^{1/2}x + \sqrt{1-\mathrm{e}^{-2t}}\Sigma^{1/2}\psi\right) = \sum_{j=1}^n \partial_j f\left(\mathrm{e}^{-t}\Sigma^{1/2}x + \sqrt{1-\mathrm{e}^{-2t}}\Sigma^{1/2}\psi\right)(\Sigma^{1/2})_{i,j}\mathrm{e}^{-t},$$

and thus

$$\begin{aligned}&\nabla f\left(\mathrm{e}^{-t}\Sigma^{1/2}x + \sqrt{1-\mathrm{e}^{-2t}}\Sigma^{1/2}\psi\right) \cdot \nabla g\left(\mathrm{e}^{-t}\Sigma^{1/2}x + \sqrt{1-\mathrm{e}^{-2t}}\Sigma^{1/2}\xi\right) \\ &= \sum_{i=1}^n\sum_{k=1}^n\sum_{j=1}^n \partial_j f\left(\mathrm{e}^{-t}\Sigma^{1/2}x + \sqrt{1-\mathrm{e}^{-2t}}\Sigma^{1/2}\psi\right) \partial_k g\left(\mathrm{e}^{-t}\Sigma^{1/2}x + \sqrt{1-\mathrm{e}^{-2t}}\Sigma^{1/2}\psi\right)(\Sigma^{1/2})_{i,j}(\Sigma^{1/2})_{i,k}\mathrm{e}^{-2t} \\ &= \sum_{k=1}^n\sum_{j=1}^n \partial_j f\left(\mathrm{e}^{-t}\Sigma^{1/2}x + \sqrt{1-\mathrm{e}^{-2t}}\Sigma^{1/2}\psi\right) \partial_k g\left(\mathrm{e}^{-t}\Sigma^{1/2}x + \sqrt{1-\mathrm{e}^{-2t}}\Sigma^{1/2}\psi\right)\mathrm{e}^{-2t}\sum_{i=1}^n(\Sigma^{1/2})_{i,j}(\Sigma^{1/2})_{i,k} \\ &= \sum_{k=1}^n\sum_{j=1}^n \partial_j f\left(\mathrm{e}^{-t}\Sigma^{1/2}x + \sqrt{1-\mathrm{e}^{-2t}}\Sigma^{1/2}\psi\right) \partial_k g\left(\mathrm{e}^{-t}\Sigma^{1/2}x + \sqrt{1-\mathrm{e}^{-2t}}\Sigma^{1/2}\psi\right)\mathrm{e}^{-2t}\Sigma_{kj},\end{aligned}$$

since $\Sigma^{1/2}$ is symmetric. The result follows: all terms are positive since f, g are coordinate-wise increasing, whence their partial derivatives are all positive, and all entries of Σ are positive.

Problem 2.12 (Local Poincaré inequalities I)**.** *Let P_t be a Markov semigroup with generator $\mathcal{L}$. For the purposes of this problem, we do not assume the existence of a stationary measure.*

a. *Prove the following variance identity:*

$$P_t(f^2) - (P_t f)^2 = 2\int_o^t P_{t-s}\Gamma(P_s f, P_s f)ds.$$

b. *Suppose that we can prove an identity of the form*

$$\Gamma(P_s f, P_s f) \le \alpha(s) P_s \Gamma(f,f)$$

for some function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$. Conclude that

$$P_t(f^2) - (P_t f)^2 \le c(t) P_t \Gamma(f,f),$$

where $c(t) = \int_0^t 2\alpha(s)ds$. Such an inequality is called a local Poincaré inequality*.*

c. *Let $(W_t)_{t\in\mathbb{R}_+}$ be standard Brownian motion. Compute an explicit expression for its semigroup and generator, and show that in this case*

$$\Gamma(P_t f, P_t f) \le P_t \Gamma(f,f).$$

Show that the local Poincaré inequality consequently provides an alternative proof of the Gaussian Poincaré inequality using Brownian motion.

d. *Let ν be a positive measure on $\mathbb{R}$ such that $\int_{\mathbb{R}}(1\wedge|x|)\nu(dx) < \infty$, and let X be an infinitely divisible random variable whose characteristic function has the Lévy-Khintchin representation $\mathbf{E}[e^{iux}] = \exp\left\{\int(e^{iuz}-1)\nu(dz)\right\}$. Then, $X \sim X_1$, where $(X_t)_{t\in\mathbb{R}_+}$ is the Lévy process with Lévy measure ν. The latter is Markov with generator*

$$\mathcal{L}f(x) = \int D_y f(x)\nu(dy), \quad \text{where } D_y f(x) := f(x+y) - f(x).$$

Use the above machinery to prove the following Poincaré inequality:

$$\mathrm{Var}[f(X)] \le \mathbf{E}\left[\int (D_y f(X))^2 \nu(dy)\right].$$

In particular, deduce Poincaré inequalities for the Poisson distribution and the one-sided exponential distribution (the latter being distinct from both Poincaré inequalities deduced in Problem 2.9).

Solution. a. Consider

$$\begin{aligned}\frac{d}{ds}P_{t-s}(P_s f)^2 &= -\mathcal{L}P_{t-s}(P_s f)^2 + 2P_{t-s}\left[(P_s f)(\mathcal{L}P_s f)\right]\\ &= -P_{t-s}\left[\mathcal{L}(P_s f)^2 - 2(P_s f)(\mathcal{L}P_s f)\right] = -2P_{t-s}\Gamma(P_s f, P_s f)\end{aligned}$$

Integrating both from $0 \le s \le t$ gives

$$\int_0^t \frac{d}{ds}P_{t-s}(P_s f)^2 ds = (P_t f)^2 - P_t(f^2) = -2\int_0^t P_{t-s}\Gamma(P_s f, P_s f)ds$$

as required.

b. Applying the inequality $\Gamma(P_s f, P_s f) \le \alpha(s)P_s\Gamma(f,f)$ to equality of previous, gives

$$P_t(f^2) - (P_t f)^2 = 2\int_0^t P_{t-s}\Gamma(P_s f, P_s f)ds \le 2\int_0^t \alpha(s)P_{t-s}P_s\Gamma(f,f)ds = P_t\Gamma(f,f)\int_0^t 2\alpha(s)ds = P_t\Gamma(f,f)c(t) \tag{1}$$

as required.

c. Brownian Motion has the following semigroup

$$P_t f(x) = E[f(x + \sqrt{t}\epsilon)] \quad \epsilon \sim N(0,1)$$

with generator derived from

$$\begin{aligned}\mathcal{L}P_t f(x) &= \frac{d}{dt}E[f(x+\sqrt{t}\epsilon)] \\ &= E[f'(x+\sqrt{t}\epsilon)\frac{\epsilon}{2\sqrt{t}}] = E[\frac{1}{2}f''(x+\sqrt{t}\epsilon)] = P_t\frac{1}{2}f''(x)\end{aligned}$$

implying $\mathcal{L}f(x) = \frac{1}{2}f''(x)$. The inner coming from applying $E[\epsilon g(\epsilon)] = E[g'(\epsilon)]$, with $g(\epsilon) = f'(x+\sqrt{t}\epsilon)$ and $g'(\epsilon) = f''(x+\sqrt{t}\epsilon)\sqrt{t}$. Noting that $\mathcal{L}(f^2) = (f')^2 + ff''$ we see that

$$\Gamma(f,f) = \frac{1}{2}\left[\mathcal{L}(f^2) - 2f\mathcal{L}f\right] = \frac{1}{2}(f')^2.$$

Therefore while noting that $\frac{d}{dx}P_t f(x) = P_t f'(x)$, we have

$$\Gamma(P_t f, P_t f) = \frac{1}{2}\left[(P_t f)'\right]^2 = \frac{1}{2}\left(P_t f'\right)^2 \leq \frac{1}{2}P_t(f')^2 = P_t\Gamma(f,f)$$

with the inequality coming from the convexity of x^2. Therefore we can apply local Poincare inequality with $\alpha(s) = 1$, and thus $c(t) = 2t$, giving

$$P_t(f^2) - (P_t f)^2 \leq tP_t\left[(f')^2\right].$$

Next note we can rescale Brownian motion to obtain a $N(0,1)$ random variable. Letting h be some integrable function under $N(0,1)$ measure, then we can state

$$Var(h(\epsilon)) = Var\left(h\left(\frac{B_t}{\sqrt{t}}\right)\right)$$

where B_t is Brownian motion. Therefore substituting $f(x) = h\left(\frac{x}{\sqrt{t}}\right)$ and $f'(x) = \frac{1}{\sqrt{t}}h'\left(\frac{x}{\sqrt{t}}\right)$ into the local Poincare inequalities of previous will give

$$\begin{aligned}Var(h(\epsilon)) &= E[h(\epsilon)^2] - E^2[h(\epsilon)] \\ &= E[f(B_t)^2] - E^2[f(B_t)] \\ &= P_t(f^2) - (P_t f)^2 \leq tP_t\left((f')^2\right) = tE[(f'(B_t))^2] = E\left[h'\left(\frac{B_t}{\sqrt{t}}\right)^2\right] = E[h'(\epsilon)^2]\end{aligned}$$

as required.

d.

Problem 2.13 (Local Poincaré inequalities II)**.** *The approach of Problem 2.12 makes it possible to obtain Poincaré inequalities using Markov processes that do not admit a stationary measure. However, even for ergodic Markov processes, it can be useful to develop a Poincaré inequality for the stationary measure μ by letting $t \to \infty$ in a local Poincaré inequality. The reason for this is the following result that will be proved in this problem.*

Theorem 1. The following are equivalent:

1. $c\Gamma_2(f,f) \geq \Gamma(f,f)$ for all f (Bakry-Émery criterion).
2. $\Gamma(P_t f, P_t f) \leq e^{-2t/c} P_t \Gamma(f,f)$ for all f, t (local ergodicity).
3. $P_t(f^2) - (P_t f)^2 \leq c(1-e^{-2t/c}) P_t \Gamma(f,f)$ for all f, t (local Poincaré).

Here we defined

$$\Gamma_2(f,g) := \frac{1}{2}\left\{\mathcal{L}\Gamma(f,g) - \Gamma(f,\mathcal{L}g) - \Gamma(\mathcal{L}f,g)\right\}.$$

This is called the iterated carré du champ *or* Γ_2-operator. *(Further discussion is provided in the notes, page 43).*

Let us prove the various implications of the above theorem.

a. *Prove 2 ⇒ 3. Hint: this follows easily as in Problem 2.12.*

b. *Prove 1 ⇒ 2. Hint:* $\frac{d}{ds}P_{t-s}\Gamma(P_s f, P_s f)$.

c. *Prove 3 ⇒ 1. Hint:* $\lim_{t\downarrow 0} t^{-2}\left\{P_t(f^2) - (P_t f)^2 - c(1-e^{-2t/c})P_t\Gamma(f,f)\right\}$.

d. *This part is too long and is given in the notes.*

Solution. a. By assumption, we have $\Gamma(P_t f, P_t f) \leq e^{-2t/c} P_t \Gamma(f,f)$ for all f and t. Now, since $2\int_0^t e^{-2s/c}\mathrm{d}s = c(1-e^{-2t/c})$, then by Problem 2.12 (b) we have that

$$P_t(f^2) - (P_t f)^2 \leq c(1-e^{-2t/c}) P_t \Gamma(f,f).$$

for all f and t.

b. By the chain rule, we have that

$$\begin{aligned}
\frac{d}{ds}P_{t-s}\Gamma(P_s f, P_s f) &= \frac{d}{ds}P_{t-s}\left(\frac{1}{2}\mathcal{L}\left((P_s f)^2\right) - P_s f \mathcal{L}(P_s f)\right) \\
&= -\mathcal{L}P_{t-s}\left(\frac{1}{2}\mathcal{L}\left((P_s f)^2\right) - P_s f \mathcal{L}(P_s f)\right) + \\
&\quad P_{t-s}\left(\frac{1}{2}\mathcal{L}\left(2P_s f\frac{d}{ds}(P_s f)\right) - \frac{d}{ds}P_s f\mathcal{L}(P_s f) - P_s f\frac{d}{ds}\left(\mathcal{L}(P_s f)\right)\right) \\
&= P_{t-s}\Big(-\mathcal{L}\Gamma(P_s f, P_s f) + \mathcal{L}\left(P_s f \mathcal{L}(P_s f)\right) - \left(\mathcal{L}(P_s f)\right)^2 - \left(P_s f \mathcal{L}^2(P_s f)\right)\Big) \\
&= P_{t-s}\left(-\mathcal{L}\Gamma(P_s f, P_s f) + 2\Gamma\left(P_s f, \mathcal{L}(P_s f)\right)\right) \\
&= -2P_{t-s}\Gamma_2\left(P_s f, P_s f\right) \\
&\leq \frac{-2}{c}P_{t-s}\Gamma(P_s f, P_s f),
\end{aligned}$$

where the inequality follows from assuming that condition (1) in the above theorem is true. Now, since $\Gamma(f,f) \geq 0$ for all f, we have that $P_{t-s}\Gamma(f,f) \geq 0$, which implies that

$$\frac{\frac{d}{ds}P_{t-s}\Gamma(P_s f, P_s f)}{P_{t-s}\Gamma(P_s f, P_s f)} \leq \frac{-2}{c}$$

for all f, which we can also write as

$$\frac{d}{ds}\log P_{t-s}\Gamma(P_s f, P_s f) \leq \frac{-2}{c}.$$

Then, integrating both sides from $s = 0$ to t, we have that

$$\log \Gamma(P_t f, P_t f) - \log P_t \Gamma(f,f) \le \frac{-2t}{c},$$

which finally implies that

$$\Gamma(P_t f, P_t f) \le e^{-2t/c} P_t \Gamma(f,f).$$

c. We have from 2.12(a)

$$\begin{aligned}
0 &\ge P_t(f^2) - (P_t f)^2 - c(1-e^{-2t/c})P_t\Gamma(f,f) \\
&= 2\int_{s=0}^{t} P_{t-s}\Gamma(P_s f, P_s f)\mathrm{d}s - c(1-e^{-2t/c})P_t\Gamma(f,f) \\
&= 2\int_{s=0}^{t} \left[P_t\Gamma(f,f) + \int_{u=0}^{s} \frac{\mathrm{d}}{\mathrm{d}u} P_{t-u}\Gamma(P_s f, P_s f)\mathrm{d}u\right]\mathrm{d}s - c(1-e^{-2t/c})P_t\Gamma(f,f)
\end{aligned}$$

and from part (b)

$$\begin{aligned}
&= 2tP_t\Gamma(f,f) - 4\int_{s=0}^{t}\int_{u=0}^{s} P_{t-u}\Gamma_2(P_u f, P_u f)\mathrm{d}u\mathrm{d}s - c(1-e^{-2t/c})P_t\Gamma(f,f) \\
&= 2tP_t\Gamma(f,f) - 4\int_{s=0}^{t}\int_{u=0}^{s} \left[P_t\Gamma_2(f,f) + \int_{r=0}^{u}(\cdots)\mathrm{d}r\right]\mathrm{d}u\mathrm{d}s - c(1-e^{-2t/c})P_t\Gamma(f,f) \\
&= 2tP_t\Gamma(f,f) - 4P_t\Gamma_2(f,f)\frac{t^2}{2} - 2tP_t\Gamma(f,f) + c\frac{1}{2}\left(\frac{2t}{c}\right)^2 P_t\Gamma(f,f) + o(t^2)
\end{aligned}$$

after cancelling and and expanding the remaining P_t terms

$$= \frac{2}{c}\Gamma(f,f)t^2 - 2\Gamma_2(f,f)t^2 + o(t^2).$$

After dividing by t^2 and taking limits we conclude that $\frac{\Gamma(f,f)}{c} - \Gamma_2(f,f) \le 0$ or equivalently that $\Gamma(f,f) \le c\Gamma_2(f,f)$.

d. Let us first compute $\Gamma(f,f)$

$$\begin{aligned}
2\Gamma(f,f) &= \mathcal{L}(f^2) - 2f\mathcal{L}f \\
&= -\sum \frac{\partial W}{\partial x_i} 2f \frac{\partial f}{\partial x_i} + 2\sum\left[f\frac{\partial^2 f}{\partial x_i^2} + \left(\frac{\partial f}{\partial x_i}\right)^2\right] - 2f\mathcal{L}f \\
&= 2\|\nabla f\|^2 \\
\Gamma(f,f) &= \|\nabla f\|^2.
\end{aligned}$$

Then

$$\begin{aligned}
\Gamma_2(f,f) &:= \frac{1}{2}\left\{\mathcal{L}\Gamma(f,f) - 2\Gamma(f,\mathcal{L}f)\right\} \\
&= \frac{1}{2}\mathcal{L}\Gamma(f,f) - \Gamma(f,\mathcal{L}f).
\end{aligned}$$

First we compute

$$\begin{aligned}
\mathcal{L}\Gamma(f,f) &= \mathcal{L}\|\nabla f\|^2 \\
&= -\sum_{i=1}^{n} \frac{\partial W}{\partial x_i}\frac{\partial}{\partial x_i}\left[\sum_{j=1}^{n}\left(\frac{\partial f}{\partial x_j}\right)^2\right] + \sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}\left[\sum_{j=1}^{n}\left(\frac{\partial f}{\partial x_j}\right)^2\right] \\
&= -2\sum_{i,j}\frac{\partial W}{\partial x_i}\frac{\partial f}{\partial x_j}\frac{\partial^2 f}{\partial x_i \partial x_j} + 2\sum_{i,j}\left[\left(\frac{\partial^2 f}{\partial x_i\partial x_j}\right)^2 + \frac{\partial f}{\partial x_j}\frac{\partial^3 f}{\partial^2 x_i \partial x_j}\right]
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{L}(f\mathcal{L}f) &= -\sum_{i=1}^{n}\frac{\partial W}{\partial x_i}\frac{\partial}{\partial x_i}\{f\mathcal{L}f\} + \sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}[f\mathcal{L}f] \\
&= -\mathcal{L}f\sum_i \frac{\partial W}{\partial x_i}\frac{\partial f}{\partial x_i} - f\sum_i\frac{\partial W}{\partial x_i}\frac{\partial \mathcal{L}f}{\partial x_i} \\
&\quad + \sum_{i=1}^{n}\left[f\frac{\partial^2\mathcal{L}f}{\partial x_i^2} + 2\frac{\partial f}{\partial x_i}\frac{\partial \mathcal{L}f}{\partial x_i} + \mathcal{L}f\frac{\partial^2 f}{\partial x_i^2}\right] \\
&= [\mathcal{L}f]^2 + f\mathcal{L}^2 f + 2\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\frac{\partial \mathcal{L}f}{\partial x_i}
\end{aligned}$$

whence it follows that

$$\begin{aligned}
\Gamma(f,\mathcal{L}f) &= \frac{1}{2}\left[\mathcal{L}(f\mathcal{L}f) - (\mathcal{L}f)^2 - f\mathcal{L}^2 f\right] \\
&= \frac{1}{2}\left[[\mathcal{L}f]^2 + f\mathcal{L}^2 f + 2\langle\nabla f, \nabla\mathcal{L}f\rangle - (\mathcal{L}f)^2 - f\mathcal{L}^2 f\right] \\
&= \langle\nabla f, \nabla\mathcal{L}f\rangle \\
\Gamma(f,\mathcal{L}f) &= -\sum_{i,j}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\frac{\partial^2 W}{\partial x_i\partial x_j} - \sum_{i,j}\frac{\partial W}{\partial x_i}\frac{\partial f}{\partial x_i}\frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{i,j}\frac{\partial f}{\partial x_j}\frac{\partial^3 f}{\partial x_i^2\partial x_j}.
\end{aligned}$$

Therefore

$$\begin{aligned}
2\Gamma_2(f,f) &= \mathcal{L}\Gamma(f,f) - 2\Gamma(f,\mathcal{L}f) \\
&= -2\sum_{i,j}\frac{\partial W}{\partial x_i}\frac{\partial f}{\partial x_j}\frac{\partial^2 f}{\partial x_i\partial x_j} + 2\sum_{i,j}\left[\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)^2 + \frac{\partial f}{\partial x_j}\frac{\partial^3 f}{\partial^2 x_i\partial x_j}\right] \\
&\quad + 2\sum_{i,j}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\frac{\partial^2 W}{\partial x_i \partial x_j} + 2\sum_{i,j}\frac{\partial W}{\partial x_i}\frac{\partial f}{\partial x_i}\frac{\partial^2 f}{\partial x_i\partial x_j} - 2\sum_{i,j}\frac{\partial f}{\partial x_j}\frac{\partial^3 f}{\partial x_i^2 \partial x_j}
\end{aligned}$$

and after cancellations

$$\begin{aligned}
&= 2\sum_{i,j}\left(\frac{\partial^2 f}{\partial x_i\partial x_j}\right)^2 + 2\sum_{i,j}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\frac{\partial^2 W}{\partial x_i\partial x_j} \\
&= 2\langle\nabla f, \nabla f\rangle + 2\langle\nabla f, \Delta W\nabla f\rangle \\
&\geq 2\langle\nabla f, \Delta W\nabla f\rangle \\
&\geq 2\rho\|\nabla f\|^2 \\
\Gamma_2(f,f) &\geq \rho\Gamma(f,f),
\end{aligned}$$

by the ρ-uniform convexity assumption.