Probability in High Dimensions

Solutions to Exercises, week 3

August 1, 2020

2.3 Poincaré Inequalities

Problem 1.1 (Carré du champ)**.** *The carré du champ is a notion of "square gradient" in the study of Poincaré inequalities. It is defined as*

$$\Gamma(f,g) \coloneqq \frac{1}{2}\left\{\mathcal{L}(fg) - f\mathcal{L}g - g\mathcal{L}f\right\}$$

in terms of the generator $\mathcal{L}$ of a Markov process with stationary measure μ.

a. *Show that $\mathcal{E}(f,f) = \int \Gamma(f,f)d\mu$, and that $\mathcal{E}(f,g) = \int \Gamma(f,g)d\mu$ if the Markov process is also reversible.*

b. *Show that $\Gamma(f,f) \geq 0$, so that is can indeed be interpreted as a square.*

c. *Prove the Cauchy-Schwarz inequality $\Gamma(f,g)^2 \leq \Gamma(f,f)\Gamma(g,g)$.*

d. *Compute the carré du champ in the various examples of Poincaré inequalities encountered in this chapter, and convince yourself that it should indeed be interpreted as the appropriate notion of "square gradient" in each case.*

Solution. a. First note that $\mu(\mathcal{L}f^2) = 0$ for every f such that $f^2 \in \mathrm{Dom}(\mathcal{L})$ (we will not concern ourselves with domain issues here). Thus,

$$\begin{aligned}
\mathcal{E}(f,f) &= -\mu(f\mathcal{L}f) \\
&= \frac{1}{2}\mu(\mathcal{L}f^2) - \mu(f\mathcal{L}f) \\
&= \mu\left(\frac{1}{2}\mathcal{L}f^2 - f\mathcal{L}f\right) \\
&= \mu\left(\Gamma(f,f)\right) = \int \Gamma(f,f)d\mu.
\end{aligned}$$

Also, if the Markov Process is reversible,

$$\begin{aligned}
\int \Gamma(f,g)d\mu &= \frac{1}{2}\left[\mu(\mathcal{L}(fg)) - \mu(f\mathcal{L}g) - \mu(g\mathcal{L}f)\right] \\
&= \frac{-1}{2}\left[\mu(f\mathcal{L}g) + \mu(g\mathcal{L}f)\right] \\
&= \frac{-1}{2}\left[\langle f,\mathcal{L}g\rangle_\mu + \langle g,\mathcal{L}f\rangle_\mu\right] \\
&= \frac{-1}{2}\left[\langle f,\mathcal{L}g\rangle_\mu + \langle \mathcal{L}g,f\rangle_\mu\right] \\
&= -\langle f,\mathcal{L}g\rangle_\mu = \mathcal{E}(f,g).
\end{aligned}$$

b. By 2.6 (c), since $\phi(x) = x^2$ is convex, $\mathcal{L}f^2 \geq 2f\mathcal{L}f$. Thus,

$$\Gamma(f, f) = \frac{1}{2}\mathcal{L}f^2 - f\mathcal{L}f \geq 0.$$

c. We have the following for any $t \in \mathbb{R}$:

$$\begin{aligned}
\Gamma(f + tg, f + tg) &= \frac{1}{2}\mathcal{L}\left(f^2 + 2tfg + t^2 g^2\right) - (f + tg)\mathcal{L}(f + tg) \\
&= \left(\frac{1}{2}\mathcal{L}f^2 - f\mathcal{L}f\right) + t\left(\mathcal{L}(fg) - f\mathcal{L}g - g\mathcal{L}f\right) + t^2\left(\frac{1}{2}\mathcal{L}g^2 - g\mathcal{L}g\right) \\
&= \Gamma(f, f) + 2\Gamma(f, g)t + \Gamma(g, g)t^2.
\end{aligned}$$

But $\Gamma(f + tg, f + tg) \geq 0$ for any f, g and t. Thus, the above quadratic in t must also be bounded below by 0, which implies that its discriminant must be bounded above by 0. In other words,

$$\left(2\Gamma(f, g)\right)^2 - 4\Gamma(f, f)\Gamma(g, g) \leq 0,$$

which implies that

$$\Gamma(f, g)^2 \leq \Gamma(f, f)\Gamma(g, g).$$

d. For the Ornstein-Uhlenbeck process, we have that $\mathcal{L}f(x) = -xf'(x) + f''(x)$. Thus,

$$\begin{aligned}
\Gamma(f, f)(x) &= \frac{1}{2}\left(\mathcal{L}f^2(x) - 2f(x)\mathcal{L}f(x)\right) \\
&= -xf(x)f'(x) + f'(x)^2 + f(x)f''(x) + xf(x)f'(x) - f(x)f''(x) \\
&= f'(x)^2.
\end{aligned}$$

Clearly, this is the square gradient in one dimension.

Problem 1.2 (Gaussian Poincaré inequalities)**.** *The goal of this problem is to develop some simple consequences and insights for the Gaussian Poincaé inequality.*

a. *Let $X_1, ..., X_n$ be i.i.d. standard Gaussians. Show that if f is $L-$Lipschitz then* $\mathrm{Var}\left[f\left(X\right)\right] \leq L^2$*.*

b. *Let $X \sim \mathcal{N}\left(0, \Sigma\right)$ then prove the identity*

$$\mathrm{Var}\left[\max_{i=1,\ldots,n} X_i\right] \leq \max_{i=1,\ldots,n} \mathrm{Var}\left[X_i\right].$$

c. *Let ε_{ji} be i.i.d. symmetric Bernoulli variables. By the central limit theorem,*

$$f\left(\frac{1}{\sqrt{k}}\sum_{i=1}^{k}\varepsilon_{1i}, \ldots, \frac{1}{\sqrt{k}}\sum_{i=1}^{k}\varepsilon_{ni}\right) \implies f(X_1, \ldots, X_n)$$

in distribution as $k \to \infty$ when f is a bounded continuous function and $X_1, \ldots, X_n$ are i.i.d. standard Gaussians. Apply the bounded difference inequality to the left-hand side and use Taylor expansion to provide an alternative proof of the Gaussian Poincaré inequality of Corollary 2.27.

Solution. a. The Gaussian Poincaré inequality yields that

$$\mathrm{Var}\left[f\left(X\right)\right] \le \mathbb{E}\left[\left\|\nabla f\left(X\right)\right\|^2\right]$$

but for any x, y

$$\left|f\left(x\right) - f\left(y\right)\right| \le L \left\|x - y\right\|$$

so $\left\|\nabla f\left(x\right)\right\| \le L$ for all x and the result follows.

b. We use the fact that we can rewrite $X = \Sigma^{1/2} Y$ where $Y \sim \mathcal{N}\left(0, I_n\right)$. We have

$$\left|\max_{i=1,\ldots,n} X_i - \max_{i=1,\ldots,n} X_i'\right| \le \left\|X - X'\right\| = \left\|\Sigma^{1/2}\left(Y - Y'\right)\right\|$$

but

$$\left\|\Sigma^{1/2}\left(Y - Y'\right)\right\| \le \left\|\Sigma^{1/2}\right\|_2 \left\|Y - Y'\right\|.$$

We know that

$$\left\|A\right\|_2 = \sqrt{\lambda_{\max}\left(A^* A\right)}$$

so

$$\left\|\Sigma^{1/2}\right\|_F = \sqrt{\lambda_{\max}\left(\Sigma\right)} = \sqrt{\max_{i=1,\ldots,n} \mathrm{Var}\left[X_i\right]}.$$

We thus have

$$\left|\max_{i=1,\ldots,n} X_i - \max_{i=1,\ldots,n} X_i'\right| := \left|f\left(Y\right) - f\left(Y'\right)\right| \le \sqrt{\max_{i=1,\ldots,n} \mathrm{Var}\left[X_i\right]} \left\|Y - Y'\right\|.$$

We can then conclude by the result of question a).

c.